

A generalization of the Mehta-Wang determinant and Askey-Wilson polynomials

Masao ISHIKAWA^{*1}, Hiroyuki TAGAWA^{† 2}, and Jiang ZENG³

¹Department of Mathematics, Faculty of Education, University of the Ryukyus,
Nishihara, Okinawa 901-0213, Japan, ishikawa@edu.u-ryukyu.ac.jp

²Department of Mathematics, Faculty of Education, Wakayama University,
Sakaedani, Wakayama 640-8510, Japan, tagawa@math.edu.wakayama-u.ac.jp

³Institut Camille Jordan, Université Claude Bernard Lyon 1, 69622 Villeurbanne
cedex, France, zeng@math.univ-lyon1.fr

2010 Mathematics Subject Classification : Primary 05A30
Secondary 05A15, 15A15, 33D45.

Keywords : The Mehta-Wang determinants, the moments of the little q -Jacobi polynomials, the Askey-Wilson polynomials.

Abstract

Motivated by the Gaussian symplectic ensemble, Mehta and Wang evaluated the $n \times n$ determinant $\det((a + j - i)\Gamma(b + j + i))$ in 2000. When $a = 0$, Ciucu and Krattenthaler computed the associated Pfaffian $\text{Pf}((j - i)\Gamma(b + j + i))$ with an application to the two dimensional dimer system in 2011. Recently we have generalized the latter Pfaffian formula with a q -analogue by replacing the Gamma function by the moment sequence of the little q -Jacobi polynomials. On the other hand, Nishizawa has found a q -analogue of the Mehta-Wang formula. Our purpose is to generalize both the Mehta-Wang and Nishizawa formulae by using the moment sequence of the little q -Jacobi polynomials. It turns out that the corresponding determinant can be evaluated explicitly in terms of the Askey-Wilson polynomials.

1 Introduction and the main results

Motivated by the Gaussian symplectic ensemble, Mehta and Wang [14] obtain the determinant identity

$$\det((a + j - i)\Gamma(b + i + j))_{0 \leq i, j \leq n-1} = D_n \prod_{i=0}^{n-1} i! \Gamma(b + i), \quad (1.1)$$

^{*}Research partially supported by CNRS, Institut Camille Jordan, UMR 5208.

[†]Partially supported by Grant-in-Aid for Scientific Research (C) 23540017.

where (N.B. the binomial coefficient $\binom{n}{k}$ is missing in [14, (7)])

$$D_n = \sum_{k=0}^n (-1)^k \binom{n}{k} \left(\frac{b-a}{2} \right)_k \left(\frac{a+b}{2} \right)_{n-k}, \quad (1.2)$$

where $(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ is known as the *rising factorial*. This D_n satisfies the three term recurrence relation

$$D_{-1} = 0, \quad D_0 = 1, \quad D_{n+1} = aD_n + n(b+n-1)D_{n-1}, \quad (1.3)$$

which can be considered as the recurrence relation for a special case of the Meixner-Pollaczek polynomials (see [14, 15]), and one may notice that the sequence $\{\Gamma(b+n)\}_{n \geq 0}$ of the Gamma functions in the left-hand side can be considered as the moment sequence of the Laguerre polynomials (see, for example, [10, 11, 17]). Nishizawa [15] obtains a q -analogue of (1.1), which will be stated below. Here we replace the Gamma functions by the moments of the little q -Jacobi polynomials and show that we obtain a special case of the Askey-Wilson polynomials as D_n , which also generalize the two results in our previous papers [6, 7]. Before we describe our results we need more notation.

Throughout this paper we use the standard notation for q -series (see [1, 4, 10, 11]):

$$(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k), \quad (a; q)_n = \frac{(a; q)_\infty}{(aq^n; q)_\infty}$$

for any integer n . Usually $(a; q)_n$ is called the *q -shifted factorial*, and we frequently use the compact notation:

$$(a_1, a_2, \dots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n.$$

The ${}_{r+1}\phi_r$ *basic hypergeometric series* is defined by

$${}_{r+1}\phi_r \left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, \dots, b_r \end{matrix}; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_n}{(q, b_1, \dots, b_r; q)_n} z^n. \quad (1.4)$$

Here we also use the q -Gamma function

$$\Gamma_q(z) = (1-q)^{1-z} \frac{(q; q)_\infty}{(q^z; q)_\infty},$$

the q -integer $[n]_q = \frac{1-q^n}{1-q}$ and the q -factorial $[n]_q! = \prod_{k=1}^n [k]_q$. By taking the limit $q \rightarrow 1$, we obtain the hypergeometric function

$${}_{r+1}F_r \left[\begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_{r+1} \\ \beta_1, \dots, \beta_r \end{matrix}; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \cdots (\alpha_{r+1})_n}{n! (\beta_1)_n \cdots (\beta_r)_n} z^n.$$

The Askey-Wilson polynomials $p_n(x)$ [4, 10, 11] satisfy the well-known recurrence relation

$$2xp_n(x) = A_np_{n+1}(x) + B_np_n(x) + C_np_{n-1}(x), \quad n \geq 0, \quad (1.5)$$

with $p_{-1}(x) = 0$, $p_0(x) = 1$, where

$$\begin{aligned} A_n &= \frac{1 - abcdq^{n-1}}{(1 - abcdq^{2n-1})(1 - abcdq^{2n})}, \\ C_n &= \frac{(1 - q^n)(1 - abq^{n-1})(1 - acq^{n-1})(1 - adq^{n-1})}{(1 - abcdq^{2n-2})(1 - abcdq^{2n-1})} \\ &\quad \times (1 - bcq^{n-1})(1 - bdq^{n-1})(1 - cdq^{n-1}), \end{aligned}$$

and

$$\begin{aligned} B_n &= a + a^{-1} - A_na^{-1}(1 - abq^n)(1 - acq^n)(1 - adq^n) \\ &\quad - C_na/(1 - abq^{n-1})(1 - acq^{n-1})(1 - adq^{n-1}). \end{aligned}$$

They have the basic hypergeometric expression

$$p_n(x; a, b, c, d; q) = \frac{(ab, ac, ad; q)_n}{a^n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{\imath\theta}, ae^{-\imath\theta} \\ ab, ac, ad \end{matrix}; q, q \right) \quad (1.6)$$

with $x = \cos \theta$, where $\imath = \sqrt{-1}$. We also use the symbol

$$\chi(A) = \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{if } A \text{ is false.} \end{cases}$$

In [6] we have proven the Hankel determinant identity

$$\begin{aligned} \det \left(\frac{(aq; q)_{i+j+r-2}}{(abq^2; q)_{i+j+r-2}} \right)_{1 \leq i, j \leq n} &= a^{\frac{n(n-1)}{2}} q^{\frac{n(n-1)(2n-1)}{6} + \frac{n(n-1)r}{2}} \\ &\quad \times \prod_{k=1}^n \frac{(q, bq; q)_{k-1} (aq; q)_{k+r-1}}{(abq^2; q)_{k+n+r-2}} \end{aligned} \quad (1.7)$$

for a positive integer n . Here

$$\mu_n = \frac{(aq; q)_n}{(abq^2; q)_n} \quad (n = 0, 1, 2, \dots)$$

is the moments of the little q -Jacobi polynomials. In our previous paper [7] we have exploited the Pfaffian identity

$$\begin{aligned} \text{Pf} \left((q^{i-1} - q^{j-1}) \frac{(aq; q)_{i+j+r-2}}{(abq^2; q)_{i+j+r-2}} \right)_{1 \leq i, j \leq 2n} \\ = a^{n(n-1)} q^{\frac{n(n-1)(4n+1)}{3} + n(n-1)r} \prod_{k=1}^{n-1} (bq; q)_{2k} \prod_{k=1}^n \frac{(q; q)_{2k-1} (aq; q)_{2k+r-1}}{(abq^2; q)_{2(k+n)+r-3}} \end{aligned} \quad (1.8)$$

for a positive integer n (see also [12, 13]).

In [15] Nishizawa has proven the q -analogue of the Mehta-Wang result:

$$\begin{aligned} & \det([a+j-i]_q \Gamma_q(b+i+j))_{0 \leq i, j \leq n-1} \\ &= q^{na+n(n-1)b/2+n(n-1)(2n-7)/6} D_{n,q} \prod_{k=0}^{n-1} [k]_q! \cdot \Gamma_q(b+k), \end{aligned} \quad (1.9)$$

where $D_{n,q}$ satisfies the recurrence relation

$$D_{-1,q} = 0, \quad D_{0,q} = 1, \quad D_{n+1,q} = q^{-a+n} [a]_q D_{n,q} + q^{-a-b} [n]_q [b+n-1]_q D_{n-1,q}. \quad (1.10)$$

Comparing this recurrence relation with the recurrence equation

$$2xQ_n(x) = Q_{n+1}(x) + (A+B)q^n Q_n(x) + (1-q^n)(1-ABq^{n-1})Q_{n-1}(x) \quad (1.11)$$

of the Al-Salam–Chihara polynomials

$$Q_n(x) = Q_n(x; A, B; q) = \frac{(AB; q)_n}{A^n} {}_3\phi_2 \left(\begin{matrix} q^{-n}, Ae^{i\theta}, Ae^{-i\theta} \\ AB, 0 \end{matrix}; q; q \right) \quad (1.12)$$

with $x = \cos \theta$ (see [10, 11]), we may remark that $D_{n,q}$ can be considered as a special case of the Al-Salam–Chihara polynomials because

$$D_{n,q} = (-i)^n q^{-\frac{a+b}{2}n} (1-q)^{-n} Q_n \left(0; q^{\frac{a+b}{2}} i, -q^{\frac{b-a}{2}} i; q \right). \quad (1.13)$$

By this observation, we can write $D_{n,q}$ explicitly as

$$D_{n,q} = \frac{(q^b; q)_n}{q^{n(a+b)}(q-1)^n} \sum_{k=0}^n q^k \frac{(q^{-n}; q)_k}{(q; q)_k} \prod_{j=0}^{k-1} \frac{1-q^{a+b+2j}}{1-q^{b+j}}. \quad (1.14)$$

One natural question we may ask is what can we obtain if we replace the q -Gamma function in the determinant of (1.9) by the moment of the little q -Jacobi polynomials. The aim of this paper is to answer this question, and we can express the determinant by the Askey-Wilson polynomials.

Theorem 1.1. Let a, b and c be parameters, and let $n \geq 1$ and r be integers. Then we have

$$\begin{aligned} & \det \left((q^{i-1} - cq^{j-1}) \frac{(aq; q)_{i+j+r-2}}{(abq^2; q)_{i+j+r-2}} \right)_{1 \leq i, j \leq n} \\ &= (-1)^n a^{\frac{n(n-3)}{2}} q^{\frac{n(n+1)(2n-5)}{6} + \frac{n(n-3)r}{2}} (abcq^{r+1}; q^2)_n \prod_{k=1}^n \frac{(q; q)_{k-1} (aq; q)_{k+r} (bq; q)_{k-2}}{(abq^2; q)_{k+n+r-2}} \\ & \quad \times {}_4\phi_3 \left(\begin{matrix} q^{-n}, a^{\frac{1}{2}} c^{\frac{1}{2}} q^{\frac{r+1}{2}}, -a^{\frac{1}{2}} c^{\frac{1}{2}} q^{\frac{r+1}{2}}, abq^{n+r} \\ aq^{r+1}, a^{\frac{1}{2}} b^{\frac{1}{2}} c^{\frac{1}{2}} q^{\frac{r+1}{2}}, -a^{\frac{1}{2}} b^{\frac{1}{2}} c^{\frac{1}{2}} q^{\frac{r+1}{2}} \end{matrix}; q, q \right) \end{aligned} \quad (1.15)$$

$$\begin{aligned} &= (-i)^n a^{\frac{n(n-2)}{2}} c^{\frac{n}{2}} q^{\frac{n(n-2)(2n+1)}{6} + \frac{n(n-2)r}{2}} \prod_{k=1}^n \frac{(q; q)_{k-1} (aq; q)_{k+r-1} (bq; q)_{k-2}}{(abq^2; q)_{k+n+r-2}} \\ & \quad \times p_n \left(0; a^{\frac{1}{2}} c^{\frac{1}{2}} q^{\frac{r+1}{2}} i, -a^{\frac{1}{2}} c^{-\frac{1}{2}} q^{\frac{r+1}{2}} i, b^{\frac{1}{2}} i, -b^{\frac{1}{2}} i; q \right). \end{aligned} \quad (1.16)$$

Remark 1.2. If we put $c = 0$ in (1.15), then we recover our previous result (1.7) easily by using the q -Chu-Vandermonde formula [4, (1.5.3)]

$${}_2\phi_1 \left(\begin{matrix} a, q^{-n} \\ c \end{matrix}; q, q \right) = \frac{(c/a; q)_n}{(c; q)_n} a^n. \quad (1.17)$$

If we put $a = q^{\alpha-1}$, $b = 0$, $c = q^\gamma$ and $r = 0$ in (1.15), then the left-hand side equals

$$\frac{q^{\frac{n(n-1)}{2}} (1-q)^{n^2}}{\{\Gamma_q(\alpha)\}^n} \det ([\gamma + j - i]_q \Gamma_q(\alpha + i + j - 2))_{1 \leq i, j \leq n}$$

because of $(q^\alpha; q)_n = (1-q)^n \cdot \frac{\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}$, and the right-hand side equals

$$(-i)^n q^{\frac{n(n-2)}{2}\alpha + \frac{n}{2}\gamma + \frac{n(n-1)(n-2)}{3}} \prod_{k=1}^n (q; q)_{k-1} (q^\alpha; q)_{k-1} \cdot Q_n(0; q^{\frac{\alpha+\gamma}{2}} i, -q^{\frac{\alpha-\gamma}{2}} i; q)$$

because of the relation $Q_n(x; A, B; q) = p_n(x; A, B, 0, 0; q)$ between the Al-Salam-Chihara polynomials and the Askey-Wilson polynomials. Hence we obtain Nishizawa's formula (1.9) as a corollary.

In Section 3 we derive the following corollary from Theorem 1.1.

Corollary 1.3. Let a, b and c be parameters, and let $n \geq 1$ and r be integers.

(i) If the size $n = 2m$ of the matrix is even, then we have

$$\begin{aligned} & \det \left((q^{i-1} - cq^{j-1}) \frac{(aq; q)_{i+j+r-2}}{(abq^2; q)_{i+j+r-2}} \right)_{1 \leq i, j \leq 2m} \\ &= a^{2m(m-1)} c^m q^{\frac{2m(m-1)(4m+1)}{3} + 2m(m-1)r} \prod_{k=1}^m \left\{ \frac{(q; q)_{2k-1} (aq; q)_{2k+r-1} (bq; q)_{2k-2}}{(abq^2; q)_{2(k+m)+r-3}} \right\}^2 \\ & \quad \times {}_4\phi_3 \left(\begin{matrix} q^{-2m}, b^{-1}q^{-2m+1}, c, c^{-1} \\ q, aq^{r+1}, a^{-1}b^{-1}q^{1-4m-r} \end{matrix}; q^2, q^2 \right) \end{aligned} \quad (1.18)$$

$$\begin{aligned} &= (-1)^m a^{m(2m-1)} b^m c^m q^{\frac{m(8m^2+3m-2)}{3} + m(2m-1)r} \prod_{k=1}^{2m} \frac{(q; q)_{k-1} (aq; q)_{k+r-1}}{(abq^2; q)_{k+2m+r-2}} \\ & \quad \times \prod_{k=1}^m \{(bq; q)_{2k-2}\}^2 \cdot p_m \left(\frac{c + c^{-1}}{2}; 1, q, aq^{r+1}, a^{-1}b^{-1}q^{1-4m-r}; q^2 \right). \end{aligned} \quad (1.19)$$

(ii) If the size $n = 2m + 1$ of the matrix is odd, then we have

$$\begin{aligned}
& \det \left((q^{i-1} - cq^{j-1}) \frac{(aq; q)_{i+j+r-2}}{(abq^2; q)_{i+j+r-2}} \right)_{1 \leq i, j \leq 2m+1} \\
&= a^{2m^2} c^m q^{\frac{2m(m+1)(4m-1)}{3} + 2m^2 r} \cdot \frac{1-c}{1-q} \cdot \prod_{k=1}^{m+1} \frac{(q; q)_{2k-1} (aq; q)_{2k+r-2} (bq; q)_{2k-2}}{(abq^2; q)_{2(k+m-1)+r}} \\
&\quad \times \prod_{k=1}^m \frac{(q; q)_{2k-1} (aq; q)_{2k+r} (bq; q)_{2k-2}}{(abq^2; q)_{2(k+m-1)+r}} \cdot {}_4\phi_3 \left(\begin{matrix} q^{-2m}, b^{-1}q^{-2m+1}, cq, c^{-1}q \\ q^3, aq^{r+2}, a^{-1}b^{-1}q^{-4m-r} \end{matrix}; q^2, q^2 \right)
\end{aligned} \tag{1.20}$$

$$\begin{aligned}
&= (-1)^m a^{m(2m+1)} b^m c^m (1-c) q^{\frac{m(8m^2+15m+4)}{3} + m(2m+1)r} \prod_{k=1}^{2m+1} \frac{(q; q)_{k-1} (aq; q)_{k+r-1}}{(abq^2; q)_{k+2m+r-1}} \\
&\quad \times \prod_{k=1}^{m+1} (bq; q)_{2k-2} \cdot \prod_{k=1}^m (bq; q)_{2k-2} \cdot p_m \left(\frac{c+c^{-1}}{2}; q, q^2, aq^{r+1}, a^{-1}b^{-1}q^{-4m-r-1}; q^2 \right).
\end{aligned} \tag{1.21}$$

Remark 1.4. If we put $c = 1$ in (1.18) for the even case, then it is clear that the ${}_4\phi_3$ sum reduces to 1, so that the determinant becomes the product which equals the square of the Pfaffian (1.8) obtained in [7]. Meanwhile, it does not suffice to prove (1.8) since it is not so trivial to take the square root of the determinant and determine the sign (see [3, 7]). If we put $c = 1$ in (1.20) for the odd case, then the factor $(1-c)$ reduces the right-hand side to 0.

If we put $a = q^\alpha$, $b = q^\beta$ and $c = q^\gamma$ and let $q \rightarrow 1$ in Theorem 1.1, then we obtain the following corollary.

Corollary 1.5. Let α , β and γ be parameters, and let $n \geq 1$ and r be integers. Then we have

$$\begin{aligned}
& \det \left((\gamma + j - i) \frac{(\alpha + 1)_{i+j+r-2}}{(\alpha + \beta + 2)_{i+j+r-2}} \right)_{1 \leq i, j \leq n} \\
&= (-2)^n \left(\frac{\alpha + \beta + \gamma + r + 1}{2} \right)_n \cdot \prod_{k=1}^n \frac{(k-1)! (\alpha + 1)_{k+r} (\beta + 1)_{k-2}}{(\alpha + \beta + 2)_{k+n+r-2}} \\
&\quad \times {}_3F_2 \left(\begin{matrix} -n, \frac{\alpha + \gamma + r + 1}{2} \\ \frac{\alpha + \beta + \gamma + r + 1}{2}, \alpha + r + 1 \end{matrix}; 1 \right)
\end{aligned} \tag{1.22}$$

$$\begin{aligned}
&= (2i)^n \prod_{k=1}^n \frac{k! (\alpha + 1)_{k+r-1} (\beta + 1)_{k-2}}{(\alpha + \beta + 2)_{k+n+r-2}} \\
&\quad \times \tilde{P}_n \left(0; \frac{\alpha + \gamma + r + 1}{2}, \frac{\beta}{2}, \frac{\alpha - \gamma + r + 1}{2}, \frac{\beta}{2} \right),
\end{aligned} \tag{1.23}$$

where

$$\begin{aligned} & \tilde{P}_n(x; a, b, c, d) \\ &= i^n \frac{(a+c)_n (a+d)_n}{n!} \cdot {}_3F_2 \left(\begin{matrix} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{matrix}; 1 \right) \end{aligned}$$

are the continuous Hahn polynomials (see [10, 11]).

The *Wilson polynomials* $W_n(x; \alpha, \beta, \gamma, \delta)$ [10, 11] are defined by

$$\frac{W_n(x^2; \alpha, \beta, \gamma, \delta)}{(\alpha + \beta)_n (\alpha + \gamma)_n (\alpha + \delta)_n} = {}_4F_3 \left(\begin{matrix} -n, \alpha + \beta + \gamma + \delta + n - 1, \alpha + ix, \alpha - ix \\ \alpha + \beta, \alpha + \gamma, \alpha + \delta \end{matrix}; 1 \right). \quad (1.24)$$

By the same specialization as above, we obtain the following corollary from Corollary 1.3.

Corollary 1.6. Let α , β and γ be parameters, and let $n \geq 1$ and r be integers.

(i) If the size $n = 2m$ of the matrix is even, then we have

$$\begin{aligned} & \det \left((\gamma + j - i) \frac{(\alpha + 1)_{i+j+r-2}}{(\alpha + \beta + 2)_{i+j+r-2}} \right)_{1 \leq i, j \leq 2m} \\ &= \prod_{k=1}^m \left\{ \frac{(2k-1)! (\alpha + 1)_{2k+r-1} (\beta + 1)_{2k-2}}{(\alpha + \beta + 2)_{2(k+m)+r-3}} \right\}^2 {}_4F_3 \left(\begin{matrix} -m, -\frac{\beta-1}{2} - m, \frac{\gamma}{2}, -\frac{\gamma}{2} \\ \frac{1}{2}, \frac{\alpha+r+1}{2}, -2m - \frac{\alpha+\beta+r-1}{2} \end{matrix}; 1 \right) \end{aligned} \quad (1.25)$$

$$\begin{aligned} &= (-2)^{3m} \prod_{k=1}^{2m} \frac{(k-1)! (\alpha + 1)_{k+r-1}}{(\alpha + \beta + 2)_{k+2m+r-2}} \prod_{k=1}^m (\beta + 1)_{2k-2}^2 \\ &\quad \times W_m \left(-\frac{\gamma^2}{4}; 0, \frac{1}{2}, \frac{\alpha + r + 1}{2}, -2m - \frac{\alpha + \beta + r - 1}{2} \right). \end{aligned} \quad (1.26)$$

(ii) If the size $n = 2m + 1$ of the matrix is odd, then we have

$$\begin{aligned} & \det \left((\gamma + j - i) \frac{(\alpha + 1)_{i+j+r-2}}{(\alpha + \beta + 2)_{i+j+r-2}} \right)_{1 \leq i, j \leq 2m+1} \\ &= \gamma \prod_{k=1}^{m+1} \frac{(2k-1)! (\alpha + 1)_{2k+r-2} (\beta + 1)_{2k-2}}{(\alpha + \beta + 2)_{2(k+m-1)+r}} \prod_{k=1}^m \frac{(2k-1)! (\alpha + 1)_{2k+r} (\beta + 1)_{2k-2}}{(\alpha + \beta + 2)_{2(k+m-1)+r}} \\ &\quad \times {}_4F_3 \left(\begin{matrix} -m, -\frac{\beta-1}{2} - m, \frac{1+\gamma}{2}, \frac{1-\gamma}{2} \\ \frac{3}{2}, \frac{\alpha+r}{2} + 1, -2m - \frac{\alpha+\beta+r}{2} \end{matrix}; 1 \right) \end{aligned} \quad (1.27)$$

$$\begin{aligned} &= (-2)^{3m} \gamma \prod_{k=1}^{2m+1} \frac{(k-1)! (\alpha + 1)_{k+r-1}}{(\alpha + \beta + 2)_{k+2m+r-1}} \prod_{k=1}^{m+1} (\beta + 1)_{2k-2} \prod_{k=1}^m (\beta + 1)_{2k-2} \\ &\quad \times W_m \left(-\frac{\gamma^2}{4}; \frac{1}{2}, 1, \frac{\alpha + r + 1}{2}, -2m - \frac{\alpha + \beta + r + 1}{2} \right). \end{aligned} \quad (1.28)$$

This paper is composed as follows. In Section 2 we give a generalization of Theorem 1.1 which is a determinant formula for arbitrary rows (see Theorem 2.1). The most labors to prove our theorems exist in the proof of this formula. In Section 3 we prove the main theorems in this section from Theorem 2.1, which will be straightforward. In Section 4 we derive a quadratic relation among the Askey-Wilson polynomials as a corollary of Theorem 1.1.

2 Determinant formula for arbitrary rows

In our previous paper [6], we prove the following formula in which the rows are arbitrary chosen. Let n be a positive integer, and k_1, \dots, k_n be arbitrary positive integers. Then we have

$$\det \left(\frac{(aq; q)_{k_i+j-2}}{(abq^2; q)_{k_i+j-2}} \right)_{1 \leq i, j \leq n} = a^{\frac{n(n-1)}{2}} q^{\frac{(n+1)n(n-1)}{6}} \times \prod_{i=1}^n \frac{(aq; q)_{k_i-1}}{(abq^2; q)_{k_i+n-2}} \prod_{1 \leq i < j \leq n} (q^{k_i-1} - q^{k_j-1}) \prod_{j=1}^n (bq; q)_{j-1}. \quad (2.1)$$

This formula is a generalization of (1.7) and a special case is obtained in [8, Theorem 3]. In this section we give this type formula, i.e., Theorem 2.1, which is crucial to prove Theorem 1.1.

First we fix some notation. If a and b are integers, we write $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. We also write $[n] = [1, n]$ for short. If S is a finite set and r a nonnegative integer, let $\binom{S}{r}$ denote the set of all r -element subsets of S . Let A be an $m \times n$ matrix. If $\mathbf{i} = (i_1, \dots, i_r)$ is an r -tuple of positive integers and $\mathbf{j} = (j_1, \dots, j_s)$ is an s -tuple of positive integers, then let $A_{\mathbf{j}}^{\mathbf{i}} = A_{j_1, \dots, j_s}^{i_1, \dots, i_r}$ denote the submatrix formed by selecting the row \mathbf{i} and the column \mathbf{j} from A . The main aim of this section is to prove the following key theorem.

Theorem 2.1. Let a, b and c be parameters. Let n be a positive integer, and $\mathbf{k} = (k_1, \dots, k_n)$ be an n -tuple of positive integers. Then we have

$$\det \left((q^{k_i-1} - cq^{j-1}) \frac{(aq; q)_{k_i+j-2}}{(abq^2; q)_{k_i+j-2}} \right)_{1 \leq i, j \leq n} = a^{\frac{n(n-3)}{2}} q^{\frac{n(n+1)(n-4)}{6}} \prod_{i=1}^n \frac{(aq; q)_{k_i-1} (bq; q)_{i-2}}{(abq^2; q)_{k_i+n-2}} \prod_{1 \leq i < j \leq n} (q^{k_i-1} - q^{k_j-1}) \times \sum_{\nu=0}^n (-1)^{n-\nu} (abcq^{2\nu+1}; q^2)_{n-\nu} (acq; q^2)_{\nu} R_{n,\nu}(\mathbf{k}, a, b; q), \quad (2.2)$$

where

$$R_{n,\nu}(\mathbf{k}, a, b; q) = \sum_{(\mathbf{i}, \mathbf{j})} q^{\sum_{l=1}^{n-\nu} i_l - n + \nu} \times \prod_{l=1}^{n-\nu} (1 - aq^{k_{i_l} - i_l + l + \nu}) \prod_{l=1}^{\nu} (1 - abq^{k_{j_l} + j_l - l + \nu - 1}). \quad (2.3)$$

Here the sum on the right-hand side runs over all pairs (\mathbf{i}, \mathbf{j}) such that $[n]$ is a disjoint union of $\mathbf{i} = \{i_1, \dots, i_{n-\nu}\} \in \binom{[n]}{n-\nu}$ and $\mathbf{j} = \{j_1, \dots, j_\nu\} \in \binom{[n]}{\nu}$ (i.e., $\mathbf{i} \cup \mathbf{j} = [n]$ and $\mathbf{i} \cap \mathbf{j} = \emptyset$). Hereafter, we also use the convention that $R_{n,\nu}(\mathbf{k}, a, b; q)$ is 0 unless $0 \leq \nu \leq n$.

For example, if $n = 3$ and $\nu = 2$, then the pairs (\mathbf{i}, \mathbf{j}) runs over

$$\{(\{1\}, \{2, 3\}), (\{2\}, \{1, 3\}), (\{3\}, \{1, 2\})\}.$$

Hence we have

$$\begin{aligned} R_{3,2}(\{k_1, k_2, k_3\}, a, b; q) &= (1 - aq^{k_1+2})(1 - abq^{k_2+2})(1 - abq^{k_3+2}) \\ &\quad + q(1 - aq^{k_2+1})(1 - abq^{k_1+1})(1 - abq^{k_3+2}) \\ &\quad + q^2(1 - aq^{k_3})(1 - abq^{k_1+1})(1 - abq^{k_2+1}). \end{aligned}$$

Let n be a positive integer, and let a, b, c and q be parameters. For an index set $\mathbf{k} = \{k_1, \dots, k_n\}$ of positive integers, let $M_n(\mathbf{k}, a, b, c; q) = (M_n(\mathbf{k}, a, b, c; q)_{i,j})_{1 \leq i, j \leq n}$ denote the matrix whose (i, j) entry is given by

$$M_n(\mathbf{k}, a, b, c; q)_{i,j} = (q^{k_i-1} - cq^{j-1})(aq^{k_i}; q)_{j-1}(abq^{k_i+j}; q)_{n-j}. \quad (2.4)$$

Then we have

$$\det \left((q^{k_i-1} - cq^{j-1}) \frac{(aq; q)_{k_i+j-2}}{(abq^2; q)_{k_i+j-2}} \right)_{1 \leq i, j \leq n} = \prod_{i=1}^n \frac{(aq; q)_{k_i-1}}{(abq^2; q)_{k_i+n-2}} \cdot \det M_n(\mathbf{k}, a, b, c; q). \quad (2.5)$$

Hence it is enough to evaluate $\det M_n(\mathbf{k}, a, b, c; q)$ to prove Theorem 2.1. The main task of this evaluation is to show the following recurrence equation:

$$\begin{aligned} &\frac{\det M_n(\mathbf{k}, a, b, c; q)}{a^{n-2}(bq; q)_{n-2} \prod_{i=1}^{n-1} (q^{k_i} - q^{k_n})} \\ &= q^{-1}(1 - acq)(1 - abq^{k_n+n-1}) \det M_{n-1}(\mathbf{k}', aq, b, cq; q) \\ &\quad - q^{n(n-3)/2}(1 - abcq^{2n-1})(1 - aq^{k_n}) \det M_{n-1}(\mathbf{k}', a, b, c; q), \end{aligned} \quad (2.6)$$

where $\mathbf{k}' = \{k_1, \dots, k_{n-1}\}$ denote the subset of the first $(n-1)$ indices of $\mathbf{k} = \{k_1, \dots, k_{n-1}, k_n\}$. This identity enable us to prove Theorem 2.1 by induction. First we cite the q -binomial formula [4, 10, 11, 16] (see also [15, Lemma 2]).

Proposition 2.2. Let n be a nonnegative integer. Then we have

$$\sum_{k=0}^n (-1)^k x^k q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix}_q = (x; q)_n. \quad (2.7)$$

Our method to evaluate $\det M_n(\mathbf{k}, a, b, c; q)$ is completely different from Mehta-Wang's proof or Nishizawa's q -analogue. It seems very tough to generalize their proof to our case. So we establish the inductive identity (2.6) and appeal to the induction on the matrix size. For the purpose the following proposition plays an essential role in the matrix multiplication of $M_n(\mathbf{k}, a, b, c; q)$ which we use in the proof of Lemma 2.5.

Lemma 2.3. Let n be a positive integer. Let a, b, c and q be complex numbers, and x_1, \dots, x_n be variables. Then we have

$$\begin{aligned} & - \sum_{\nu=1}^n \frac{(q^{-1}x_\nu - cq^{j-1})(ax_\nu; q)_{j-1}(abq^j x_\nu; q)_{n-j}}{x_\nu(1 - ax_\nu) \prod_{l \neq \nu}^n (x_l - x_\nu)} \\ &= \frac{cq^{j-1}}{\prod_{l=1}^n x_l} + \chi(j=1) \cdot \frac{(-1)^n a^{n-1} q^{-1} (1 - acq)(bq; q)_{n-1}}{\prod_{l=1}^n (1 - ax_l)}, \end{aligned} \quad (2.8)$$

$$\begin{aligned} & - \sum_{\nu=1}^n \frac{(q^{-1}x_\nu - cq^{j-1})(ax_\nu; q)_{j-1}(abq^j x_\nu; q)_{n-j}}{x_\nu(1 - abq^{n-1}x_\nu) \prod_{l \neq \nu}^n (x_l - x_\nu)} \\ &= \frac{cq^{j-1}}{\prod_{l=1}^n x_l} - \chi(j=n) \cdot \frac{a^{n-1} q^{n(n-3)/2} (1 - abcq^{2n-1})(bq; q)_{n-1}}{\prod_{l=1}^n (1 - abq^{n-1}x_l)}. \end{aligned} \quad (2.9)$$

Proof. There are several methods to prove these identities. Here we present a simple proof by the complex analysis (see [5]). Let $F(z)$ be the meromorphic function of z defined by

$$F(z) = \frac{(q^{-1}z - cq^{j-1})(az; q)_{j-1}(abq^j z; q)_{n-j}}{z(1 - az) \prod_{l=1}^n (x_l - z)}.$$

Then $z = x_\nu$ ($\nu = 1, \dots, n$) is a pole, and its residue is

$$\text{Res}_{z=x_\nu} F(z) = - \frac{(q^{-1}x_\nu - cq^{j-1})(ax_\nu; q)_{j-1}(abq^j x_\nu; q)_{n-j}}{x_\nu(1 - ax_\nu) \prod_{l \neq \nu}^n (x_l - x_\nu)}.$$

Similarly $z = 0$ is also a pole, and its residue is

$$\text{Res}_{z=0} F(z) = - \frac{cq^{j-1}}{\prod_{l=1}^n x_l}.$$

At $z = a^{-1}$, $F(z)$ has the residue of

$$\begin{aligned} \text{Res}_{z=a^{-1}} F(z) &= - \frac{(a^{-1}q^{-1} - cq^{j-1})(1; q)_{j-1}(bq^j; q)_{n-j}}{\prod_{l=1}^n (x_l - a^{-1})} \\ &= -\chi(j=1) \cdot \frac{(a^{-1}q^{-1} - c)(bq; q)_{n-1}}{\prod_{l=1}^n (x_l - a^{-1})}. \end{aligned}$$

Finally $z = \infty$ is also a pole of $F(z)$, and $\text{Res}_{z=\infty} F(z) = - \lim_{z \rightarrow \infty} zF(z) = 0$. Since the sum of the residues of a meromorphic function on a compact Riemann surface must be zero, we obtain the desired identity (2.8). The other identities can be proven similarly. The details are left to the reader. \square

Notice that an immediate consequence of these identities are the following Vandermonde type determinants. These identities are obtained from Lemma 2.3 by expanding the determinants along the last columns.

Corollary 2.4. Let n be a positive integer, and k be an integer such that $1 \leq k \leq n$. Let a, b, c and q be parameters, and $\mathbf{x} = (x_1, \dots, x_n)$ be an n -tuple of variables.

- (i) Let $V_{n,k}(\mathbf{x}, a, b, c; q) = (V_{n,k}(\mathbf{x}, a, b, c; q)_{i,j})_{1 \leq i, j \leq n}$ be the $n \times n$ matrix defined by

$$V_{n,k}(\mathbf{x}, a, b, c; q)_{i,j} = \begin{cases} x_i^{j-1} & \text{if } 1 \leq j < n, \\ -\frac{(x_i - cq^k)(ax_i; q)_{k-1}(abq^k x_i; q)_{n-k}}{x_i(1 - ax_i)} & \text{if } j = n, \end{cases}$$

for $1 \leq i \leq n$. Then we have

$$\begin{aligned} & \frac{(-1)^{n-1} \det V_{n,k}(\mathbf{x}, a, b, c; q)}{\prod_{1 \leq i < j \leq n} (x_j - x_i)} \\ &= \frac{cq^k}{\prod_{l=1}^n x_l} + \chi(k=1) \cdot \frac{(-1)^n a^{n-1} (1 - acq)(bq; q)_{n-1}}{\prod_{l=1}^n (1 - ax_l)}. \end{aligned}$$

- (ii) Let $W_{n,k}(\mathbf{x}, a, b, c; q) = (W_{n,k}(\mathbf{x}, a, b, c; q)_{i,j})_{1 \leq i, j \leq n}$ be the $n \times n$ matrix defined by

$$W_{n,k}(\mathbf{x}, a, b, c; q)_{i,j} = \begin{cases} x_i^{j-1} & \text{if } 1 \leq j < n, \\ -\frac{(x_i - cq^k)(ax_i; q)_{k-1}(abq^k x_i; q)_{n-k}}{x_i(1 - abq^{n-1} x_i)} & \text{if } j = n, \end{cases}$$

for $1 \leq i \leq n$. Then we have

$$\begin{aligned} & \frac{(-1)^{n-1} \det W_{n,k}(\mathbf{x}, a, b, c; q)}{\prod_{1 \leq i < j \leq n} (x_j - x_i)} \\ &= \frac{cq^k}{\prod_{l=1}^n x_l} - \chi(k=n) \cdot \frac{a^{n-1} q^{(n-1)(n-2)/2} (1 - abcq^{2n-1})(bq; q)_{n-1}}{\prod_{l=1}^n (1 - abq^{n-1} x_l)}. \end{aligned}$$

Here we will not use these Vandermonde type determinants, but it seems that these identities are worth mentioning. We introduce four triangular matrices $X_n(\mathbf{k}, a; q)$, $Y_n(q)$, $L_n(\mathbf{k}, a, b; q)$ and $U_n(q)$ which plays an important role to manipulate $M_n(\mathbf{k}, a, b, c; q)$ in (2.5). Let $X_n(\mathbf{k}, a; q) = (X(\mathbf{k}, a; q)_{i,j})_{1 \leq i, j \leq n}$ and $Y_n(q) = (Y_n(q)_{i,j})_{1 \leq i, j \leq n}$ be the $n \times n$ lower triangular matrices whose (i, j) -entry is, respectively, given by

$$X(\mathbf{k}, a; q)_{i,j} = -\frac{\chi(i \geq j)}{q^{k_j} (1 - aq^{k_j}) \prod_{\substack{l=1 \\ l \neq j}}^i (q^{k_l} - q^{k_j})}, \quad (2.10)$$

$$Y_n(q)_{i,j} = (-1)^{i+j} q^{-\frac{(i-j)(2n+1-i-j)}{2}} \begin{bmatrix} n-j \\ i-j \end{bmatrix}_q. \quad (2.11)$$

Similarly, let $L_n(\mathbf{k}, a, b; q) = (L_n(\mathbf{k}, a, b; q)_{i,j})_{1 \leq i, j \leq n}$ (resp. $U_n(q) = (U(q)_{i,j})_{1 \leq i, j \leq n}$) be the $n \times n$ lower (resp. upper) triangular matrix whose (i, j) -entry is, respectively, given by

$$L_n(\mathbf{k}, a, b; q)_{i,j} = -\frac{\chi(i \geq j)}{q^{k_j}(1 - abq^{k_j+n-1}) \prod_{\substack{l=1 \\ l \neq j}}^i (q^{k_l} - q^{k_j})}, \quad (2.12)$$

$$U(q)_{i,j} = (-1)^{i+j} q^{\frac{(j-i)(j-i+1)}{2}} \begin{bmatrix} j-1 \\ j-i \end{bmatrix}_q. \quad (2.13)$$

The fact that these are all triangular matrices is very important in the following proof. But, note that these triangular matrices are irrelevant to the LU -decomposition of $M_n(\mathbf{k}, a, b, c; q)$. The LU -decomposition of $M_n(\mathbf{k}, a, b, c; q)$ seems another difficult problem. We define the $n \times n$ matrices $P_n(\mathbf{k}, a, b, c; q)$ and $Q_n(\mathbf{k}, a, b, c; q)$ by

$$\begin{aligned} P_n(\mathbf{k}, a, b, c; q) &= X_n(\mathbf{k}, a; q) M_n(\mathbf{k}, a, b, c; q) Y_n(q), \\ Q_n(\mathbf{k}, a, b, c; q) &= L_n(\mathbf{k}, a, b; q) M_n(\mathbf{k}, a, b, c; q) U_n(q). \end{aligned}$$

Since $X_n(\mathbf{k}, a; q)$, $L_n(\mathbf{k}, a, b; q)$ are triangular and $Y_n(q)$, $U_n(q)$ are unitriangular, we easily obtain

$$\det P_n(\mathbf{k}, a, b, c; q) = \frac{(-1)^n \det M_n(\mathbf{k}, a, b, c; q)}{q^{\sum_{i=1}^n k_i} \prod_{i=1}^n (1 - aq^{k_i}) \prod_{1 \leq i < j \leq n} (q^{k_i} - q^{k_j})}, \quad (2.14)$$

$$\det Q_n(\mathbf{k}, a, b, c; q) = \frac{(-1)^n \det M_n(\mathbf{k}, a, b, c; q)}{q^{\sum_{i=1}^n k_i} \prod_{i=1}^n (1 - abq^{k_i+n-1}) \prod_{1 \leq i < j \leq n} (q^{k_i} - q^{k_j})}. \quad (2.15)$$

The key to prove (2.6) will be Lemma 2.7. To prove Lemma 2.7, we need the following lemma which gives the bottom rows of $P_n(\mathbf{k}, a, b, c; q)$ and $Q_n(\mathbf{k}, a, b, c; q)$.

Lemma 2.5. Let $n \geq 2$ be an integer, and let a, b, c and q be parameters. Let $X_n(\mathbf{k}, a; q)$, $Y_n(q)$, $L_n(\mathbf{k}, a, b; q)$, $U_n(q)$, $P_n(\mathbf{k}, a, b, c; q)$ and $Q_n(\mathbf{k}, a, b, c; q)$ be as above, and $M_n(\mathbf{k}, a, b, c; q)$ as in (2.4). Then the bottom rows of $P_n(\mathbf{k}, a, b, c; q)$ and $Q_n(\mathbf{k}, a, b, c; q)$ are given by

$$P_n(\mathbf{k}, a, b, c; q)_{n,j} = \begin{cases} \frac{(-1)^n a^{n-1} q^{-1} (1-acq)(bq; q)_{n-1}}{\prod_{i=1}^n (1-aq^{k_i})} & \text{if } j = 1, \\ 0 & \text{if } 1 < j < n, \\ cq^{n-1-\sum_{i=1}^n k_i} & \text{if } j = n, \end{cases} \quad (2.16)$$

$$Q_n(\mathbf{k}, a, b, c; q)_{n,j} = \begin{cases} cq^{-\sum_{i=1}^n k_i} & \text{if } j = 1, \\ 0 & \text{if } 1 < j < n, \\ -\frac{a^{n-1} q^{n(n-3)/2} (1-abcq^{2n-1})(bq; q)_{n-1}}{\prod_{l=1}^n (1-abq^{k_l+n-1})} & \text{if } j = n. \end{cases} \quad (2.17)$$

Proof. By substituting $x_l = q^{k_l}$ for $l = 1, \dots, n$ into (2.8), we see the (n, j) -entry of $X_n(\mathbf{k}, a; q)M_n(\mathbf{k}, a, b, c; q)$ equals

$$cq^{j-1-\sum_{l=1}^n k_l} + \chi(j=1) \cdot \frac{(-1)^n a^{n-1} q^{-1} (1 - acq)(bq; q)_{n-1}}{\prod_{l=1}^n (1 - aq^{k_l})}.$$

Since $Y_n(q)$ is lower unitriangular, the $(n, 1)$ -entry of $X_n(\mathbf{k}, a; q)M_n(\mathbf{k}, a, b, c; q)$ affects only the first entry of the bottom row of $X_n(\mathbf{k}, a; q)M_n(\mathbf{k}, a, b, c; q)Y_n(q)$. Hence, if $j \neq 1$, then we have

$$P_n(\mathbf{k}, a, b, c; q)_{n,j} = \sum_{\nu=j}^n (-1)^{\nu+j} cq^{\nu-1-\frac{(\nu-j)(2n+1-\nu-j)}{2}-\sum_{l=1}^n k_l} \begin{bmatrix} n-j \\ \nu-j \end{bmatrix}_q.$$

By replacing ν by $n - \nu$, we obtain

$$P_n(\mathbf{k}, a, b, c; q)_{n,j} = cq^{j-1-\sum_{l=1}^n k_l - \frac{(n-j)(n-j-1)}{2}} \sum_{\nu=0}^{n-j} (-1)^{n-j-\nu} q^{\frac{\nu(\nu-1)}{2}} \begin{bmatrix} n-j \\ \nu \end{bmatrix}_q.$$

By (2.7), this equals $cq^{n-1-\sum_{l=1}^n k_l}$ if $j = n$, and 0 otherwise. The case where $j = 1$ is easily obtained from $Y_n(q)_{1,1} = 1$. This proves (2.16). The other identities can be proven similarly. The details are left to the reader. \square

Proposition 2.6. Let n be a positive integer, and q a parameter. Let $Y_n(q)$ and $U_n(q)$ be as in (2.11) and (2.13). Then we have

$$Y_n(q)^{-1} = \left(q^{(j-i)(n+1-i)} \begin{bmatrix} n-j \\ i-j \end{bmatrix}_q \right)_{1 \leq i, j \leq n}, \quad (2.18)$$

$$U_n(q)^{-1} = \left(q^{j-i} \begin{bmatrix} j-1 \\ i-1 \end{bmatrix}_q \right)_{1 \leq i, j \leq n}. \quad (2.19)$$

Especially, we have

$$\det Y_n(q)_{[1, n-1]}^{[1, n] \setminus \{i\}} = (-q)^{i-n}, \quad (2.20)$$

$$\det U_n(q)_{[2, n]}^{[1, n] \setminus \{i\}} = (-q)^{i-1}. \quad (2.21)$$

Proof. To prove that (2.18) gives the inverse of $Y_n(q)$, one need to show that

$$\begin{aligned} & \sum_{k=j}^i q^{(k-i)(n+1-i)} \begin{bmatrix} n-k \\ i-k \end{bmatrix}_q (-1)^{k+j} q^{-\frac{(k-j)(2n+1-k-j)}{2}} \begin{bmatrix} n-j \\ k-j \end{bmatrix}_q \\ &= (-1)^{i+j} q^{-\frac{(i-j)(2n+1-i-j)}{2}} \begin{bmatrix} n-j \\ n-i \end{bmatrix}_q \sum_{k=0}^{i-j} (-1)^k q^{\frac{k(k-1)}{2}} \begin{bmatrix} i-j \\ k \end{bmatrix}_q \end{aligned}$$

equals $\chi(i=j)$, where the second sum is obtained from the first sum by replacing $i-k$ by k . This can be shown by (2.7). It is also an easy exercise to show that (2.19) gives the inverse matrix of $U_n(q)$ using (2.7). Finally, (2.20) (resp. (2.21)) is obtained from (2.18) (resp. (2.19)) using the relation $A^{-1} = \frac{1}{\det A} \text{adj}(A)$, where the adjugate matrix $\text{adj}(A)$ is the transpose of the matrix of cofactors. \square

Let $n \leq N$ be positive integers, and let A be an $n \times N$ matrix and B an $N \times n$ matrix. Then the following formula is known as the Cauchy–Binet formula:

$$\det AB = \sum_{\mathbf{i} \in \binom{[N]}{n}} A_{\mathbf{i}}^{[n]} B_{[n]}^{\mathbf{i}}. \quad (2.22)$$

Lemma 2.7. Let n be a positive integer, and let a, b, c and q be parameters. Let $P_n(\mathbf{k}, a, b, c; q)$ and $Q_n(\mathbf{k}, a, b, c; q)$ be as defined in Lemma 2.5. When $\mathbf{k} = \{k_1, \dots, k_{n-1}, k_n\}$ is a row index set, let $\mathbf{k}' = \{k_1, \dots, k_{n-1}\}$ denote the subset of the first $(n-1)$ indices of \mathbf{k} . Then we have

$$\det P_n(\mathbf{k}, a, b, c; q)_{[2, n]}^{[1, n-1]} = \frac{(-1)^{n-1} \det M_{n-1}(\mathbf{k}', aq, b, cq; q)}{q^{\sum_{i=1}^{n-1} k_i} \prod_{1 \leq i < j < n} (q^{k_i} - q^{k_j})}, \quad (2.23)$$

$$\det Q_n(\mathbf{k}, a, b, c; q)_{[1, n-1]}^{[1, n-1]} = \frac{(-1)^{n-1} \det M_{n-1}(\mathbf{k}', a, b, c; q)}{q^{\sum_{i=1}^{n-1} k_i} \prod_{1 \leq i < j < n} (q^{k_i} - q^{k_j})}, \quad (2.24)$$

$$\frac{\det P_n(\mathbf{k}, a, b, c; q)_{[1, n-1]}^{[1, n-1]}}{\prod_{\nu=1}^{n-1} (1 - abq^{k_\nu + n-1})} = (-q)^{-n+1} \frac{\det Q_n(\mathbf{k}, a, b, c; q)_{[2, n]}^{[1, n-1]}}{\prod_{\nu=1}^{n-1} (1 - aq^{k_\nu})}. \quad (2.25)$$

Proof. Since $X_n(\mathbf{k}, a; q)$ and $Y_n(q)$ are lower triangular, we have

$$P_n(\mathbf{k}, a, b, c; q)_{[2, n]}^{[1, n-1]} = X_n(\mathbf{k}, a; q)_{[1, n-1]}^{[1, n-1]} M_n(\mathbf{k}, a, b, c; q)_{[2, n]}^{[1, n-1]} Y_n(q)_{[2, n]}^{[2, n]}.$$

Hence, (2.23) easily follows from

$$\det X_n(\mathbf{k}, a; q)_{[1, n-1]}^{[1, n-1]} = \frac{(-1)^{n-1}}{q^{\sum_{\nu=1}^{n-1} k_\nu} \prod_{\nu=1}^{n-1} (1 - aq^{k_\nu}) \prod_{1 \leq i < j < n} (q^{k_i} - q^{k_j})},$$

$$\det M_n(\mathbf{k}, a, b, c; q)_{[2, n]}^{[1, n-1]} = \prod_{\nu=1}^{n-1} (1 - aq^{k_\nu}) \cdot \det M_{n-1}(\mathbf{k}', aq, b, cq; q),$$

$$\det Y_n(q)_{[2, n]}^{[2, n]} = 1.$$

Here, the first and the third identities follow from the fact that $X_n(\mathbf{k}, a; q)$ and $Y_n(q)$ are lower triangular, and the second identity follows from the fact that (i, j) entry of $M_n(\mathbf{k}, a, b, c; q)_{[2, n]}^{[1, n-1]}$ equals $(q^{k_i-1} - cq^j)(aq^{k_i}; q)_j (abq^{k_i+j+1}; q)_{n-j-1}$ for $1 \leq i, j < n$ (see (2.4)). Exactly the same argument proves (2.24) from the fact

$$Q_n(\mathbf{k}, a, b, c; q)_{[1, n-1]}^{[1, n-1]} = L_n(\mathbf{k}, a, b; q)_{[1, n-1]}^{[1, n-1]} M_n(\mathbf{k}, a, b, c; q)_{[1, n-1]}^{[1, n-1]} U_n(q)_{[1, n-1]}^{[1, n-1]}.$$

The details of this argument are left to the reader. Finally, we prove (2.25). If there is no fear of confusion, we may write $M_n(\mathbf{k}, a, b, c; q)$ (resp. $P_n(\mathbf{k}, a, b, c; q)$, $Q_n(\mathbf{k}, a, b, c; q)$, $X_n(\mathbf{k}, a; q)$, $Y_n(q)$, $L_n(\mathbf{k}, a, b; q)$ and $U_n(q)$) as $M_n(\mathbf{k})$ (resp. $P_n(\mathbf{k})$, $Q_n(\mathbf{k})$, $X_n(\mathbf{k})$, Y_n , $L_n(\mathbf{k})$ and U_n) in short hereafter. We use the identities

$$\begin{aligned} P_n(\mathbf{k})_{[1, n-1]}^{[1, n-1]} &= X_n(\mathbf{k})_{[1, n-1]}^{[1, n-1]} M_n(\mathbf{k})_{[1, n]}^{[1, n-1]} Y_n^{[1, n]}_{[1, n-1]}, \\ Q_n(\mathbf{k})_{[2, n]}^{[1, n-1]} &= L_n(\mathbf{k})_{[1, n-1]}^{[1, n-1]} M_n(\mathbf{k})_{[1, n]}^{[1, n-1]} U_n^{[1, n]}_{[2, n]}, \end{aligned}$$

which come from the triangularities of the matrices as before. Hence, by taking the determinant of the both sides, we obtain

$$\begin{aligned}\det P_n(\mathbf{k})_{[1,n-1]}^{[1,n-1]} &= \det X_n(\mathbf{k})_{[1,n-1]}^{[1,n-1]} \cdot \det M_n(\mathbf{k})_{[1,n]}^{[1,n-1]} Y_n^{[1,n]}_{[1,n-1]}, \\ \det Q_n(\mathbf{k})_{[2,n]}^{[1,n-1]} &= \det L_n(\mathbf{k})_{[1,n-1]}^{[1,n-1]} \cdot \det M_n(\mathbf{k})_{[1,n]}^{[1,n-1]} U_n^{[1,n]}_{[2,n]}.\end{aligned}$$

Since the $X_n(\mathbf{k}, a; q)_{[1,n-1]}^{[1,n-1]}$ and $L_n(\mathbf{k}, a, b; q)_{[1,n-1]}^{[1,n-1]}$ are lower triangular, it is easy to compute the first determinant of the right-hand side of each equality. Applying the Cauchy–Binet formula (2.22) to the second determinant of the right-hand side of each equality, we obtain

$$\begin{aligned}\det P_n(\mathbf{k})_{[1,n-1]}^{[1,n-1]} &= \frac{(-1)^{n-1} \sum_{i=1}^n \det M_n(\mathbf{k})_{[1,n] \setminus \{i\}}^{[1,n-1]} \det Y_n^{[1,n]}_{[1,n-1] \setminus \{i\}}}{q^{\sum_{\nu=1}^{n-1} k_\nu} \prod_{\nu=1}^{n-1} (1 - aq^{k_\nu}) \prod_{1 \leq i < j < n} (q^{k_i} - q^{k_j})}, \\ \det Q_n(\mathbf{k})_{[2,n]}^{[1,n-1]} &= \frac{(-1)^{n-1} \sum_{i=1}^n \det M_n(\mathbf{k})_{[1,n] \setminus \{i\}}^{[1,n-1]} \det U_n^{[1,n]}_{[2,n] \setminus \{i\}}}{q^{\sum_{\nu=1}^{n-1} k_\nu} \prod_{\nu=1}^{n-1} (1 - abq^{k_\nu + n-1}) \prod_{1 \leq i < j < n} (q^{k_i} - q^{k_j})}.\end{aligned}$$

Hence (2.25) immediately follows from Proposition 2.6. This completes the proof. \square

Now we are in position to prove (2.6).

Proof of (2.6). Expanding $P_n(\mathbf{k})$ and $Q_n(\mathbf{k})$ along the bottom row, we obtain

$$\begin{aligned}\det P_n(\mathbf{k}) &= -\frac{a^{n-1}q^{-1}(1-acq)(bq;q)_{n-1}}{\prod_{l=1}^n(1-aq^{k_l})} \det P_n(\mathbf{k})_{[2,n]}^{[1,n-1]} \\ &\quad + cq^{n-1-\sum_{l=1}^n k_l} \det P_n(\mathbf{k})_{[1,n-1]}^{[1,n-1]},\end{aligned}\tag{2.26}$$

$$\begin{aligned}\det Q_n(\mathbf{k}) &= (-1)^{n+1}cq^{-\sum_{l=1}^n k_l} \det Q_n(\mathbf{k})_{[2,n]}^{[1,n-1]} \\ &\quad - \frac{a^{n-1}q^{n(n-3)/2}(1-abcq^{2n-1})(bq;q)_{n-1}}{\prod_{l=1}^n(1-abq^{k_l+n-1})} \det Q_n(\mathbf{k})_{[1,n-1]}^{[1,n-1]},\end{aligned}\tag{2.27}$$

from (2.16) and (2.17). Hence, substituting (2.14) and (2.23) into (2.26), (2.15) and (2.24) into (2.27), then the resulting equalities into (2.25), we obtain the desired identity. \square

Now we are in position to prove Theorem 2.1. In fact the proof is straightforward by induction.

Proof of Theorem 2.1. First, we note that, for any integers n and ν , it holds

$$\begin{aligned}R_{n,\nu}(\mathbf{k}, a, b; q) &= (1 - abq^{k_n+n-1})R_{n-1,\nu-1}(\mathbf{k}', aq, b; q) \\ &\quad + q^{n-1}(1 - aq^{k_n})R_{n-1,\nu}(\mathbf{k}', a, b; q),\end{aligned}\tag{2.28}$$

where $\mathbf{k} = \{k_1, \dots, k_{n-1}, k_n\}$ and $\mathbf{k}' = \{k_1, \dots, k_{n-1}\}$ are as before. (2.28) follows from the definition (2.3) of $R_{n,\nu}(\mathbf{k}, a, b; q)$ by considering two exclusive cases, $j_\nu = n$ or $i_{n-\nu} = n$. Now we prove the identity

$$\begin{aligned} \det M_n(\mathbf{k}, a, b, c; q) &= (-1)^n a^{\frac{n(n-3)}{2}} q^{\frac{n(n+1)(n-4)}{6}} \prod_{i=1}^n (bq; q)_{i-2} \\ &\times \prod_{1 \leq i < j \leq n} (q^{k_i-1} - q^{k_j-1}) \sum_{\nu=0}^n (-1)^\nu (abcq^{2\nu+1}; q^2)_{n-\nu} (acq; q^2)_\nu R_{n,\nu}(\mathbf{k}, a, b; q), \end{aligned} \quad (2.29)$$

by induction on n . If $n = 1$, then the left-hand side of (2.29) is trivially $q^{k_1-1} - c$ from (2.4). It is straightforward computation to check the right-hand side equals $q^{k_1-1} - c$. Assume $n > 1$ and (2.29) holds up to $(n-1)$. Using (2.6) and the induction hypothesis, we obtain

$$\begin{aligned} &\frac{\det M_n(\mathbf{k}, a, b, c; q)}{(-1)^n a^{\frac{n(n-3)}{2}} q^{\frac{n(n^2-6n-1)}{6}} \prod_{i=1}^n (bq; q)_{i-2} \prod_{1 \leq i < j \leq n} (q^{k_i} - q^{k_j})} \\ &= (1 - abq^{k_n+n-1}) \sum_{\nu=0}^{n-1} (-1)^{\nu+1} (abcq^{2\nu+3}; q^2)_{n-\nu-1} (acq; q^2)_{\nu+1} R_{n-1,\nu}(\mathbf{k}', aq, b; q) \\ &+ q^{n-1} (1 - aq^{k_n}) \sum_{\nu=0}^{n-1} (-1)^\nu (abcq^{2\nu+1}; q^2)_{n-\nu} (acq; q^2)_\nu R_{n-1,\nu}(\mathbf{k}', a, b; q). \end{aligned}$$

Replacing $\nu+1$ by ν in the first sum and applying (2.28), we establish (2.29) for n . Hence (2.29) holds for an arbitrary positive integer n . Finally, (2.5) and (2.29) immediately implies (2.2). This completes the proof of Theorem 2.1. \square

To recover (2.1) from Theorem 2.1, one can prove

$$\sum_{\nu=0}^n (-1)^{n-\nu} R_{n,\nu}(\mathbf{k}, a, b; q) = a^n q^{\frac{n(n-1)}{2} + \sum_{l=1}^n k_l} (b; q)_n \quad (2.30)$$

by induction, appealing to (2.28). If he substitutes (2.30) into (2.2), then he obtains (2.1) immediately.

3 Proof of the main theorems

The aim of this section is to derive Theorem 1.1 from Theorem 2.1, and then prove Corollary 1.3 from Theorem 1.1. Once we prove Theorem 2.1, then it is easy and straightforward to prove the main theorems mainly by induction. First, to prove Theorem 1.1, we set $\mathbf{k} = [n] = \{1, 2, \dots, n\}$ in (2.2). Hence, it is essential to prove the following lemma before proceeding to the proof of (1.16).

Lemma 3.1. If we put $\mathbf{k} = [n]$ in (2.2), then we obtain

$$R_{n,\nu}([n], a, b; q) = q^{\frac{(n-\nu)(n-\nu-1)}{2}} \begin{bmatrix} n \\ \nu \end{bmatrix}_q (aq^{\nu+1}; q)_{n-\nu} (abq^n; q)_\nu. \quad (3.1)$$

Proof. We proceed by induction on n . If $n = 0$, then the both-sides are equal to 1 if $\nu = 0$, and 0 otherwise. Let $n > 0$, and assume (3.1) holds when $n - 1$. By (2.28), the left-hand side satisfies

$$R_{n,\nu}([n], a, b; q) = (1 - abq^{2n-1}) R_{n-1,\nu-1}([n-1], aq, b; q) + q^{n-1} (1 - aq^n) R_{n-1,\nu}([n-1], a, b; q).$$

Substituting the induction hypothesis into this identity, we see the right-hand side equals

$$q^{\frac{(n-\nu)(n-\nu-1)}{2}} (aq^{\nu+1}; q)_{n-\nu} (abq^n; q)_{\nu-1} \times \left\{ \begin{bmatrix} n-1 \\ \nu-1 \end{bmatrix}_q (1 - abq^{2n-1}) + q^\nu \begin{bmatrix} n-1 \\ \nu \end{bmatrix}_q (1 - abq^{n-1}) \right\},$$

which becomes the right-hand side of (3.1). Hence this proves that (3.1) holds for an arbitrary nonnegative number n . \square

Now we are in position to prove Theorem 1.1

Proof of Theorem 1.1. If we substitute $\mathbf{k} = [n]$ into (2.2) using (3.1), then we see that $\det \left((q^{i-1} - cq^{j-1}) \frac{(aq; q)_{i+j-2}}{(abq^2; q)_{i+j-2}} \right)_{1 \leq i, j \leq n}$ equals

$$a^{\frac{n(n-3)}{2}} q^{\frac{n(n+1)(n-4)}{6}} \prod_{k=1}^n \frac{(aq; q)_{k-1} (bq; q)_{k-2}}{(abq^2; q)_{k+n-2}} \prod_{1 \leq i < j \leq n} (q^{i-1} - q^{j-1}) \times \sum_{\nu=0}^n (-1)^{n-\nu} q^{\frac{(n-\nu)(n-\nu-1)}{2}} \begin{bmatrix} n \\ \nu \end{bmatrix}_q (abcq^{2\nu+1}; q^2)_{n-\nu} (acq; q^2)_\nu (aq^{\nu+1}; q)_{n-\nu} (abq^n; q)_\nu.$$

By rewriting the sum with the basic hypergeometric series by using $\prod_{1 \leq i < j \leq n} (q^{i-1} - q^{j-1}) = q^{\frac{n(n-1)(n-2)}{6}} \prod_{k=0}^{n-1} (q; q)_k \begin{bmatrix} n \\ \nu \end{bmatrix}_q = (-1)^\nu q^{n\nu - \frac{\nu(\nu-1)}{2}} \frac{(q^{-n}; q)_\nu}{(q; q)_\nu}$, $(abcq^{2\nu+1}; q^2)_{n-\nu} = \frac{(abcq; q^2)_n}{(abcq; q^2)_\nu}$ and $(aq^{\nu+1}; q)_{n-\nu} = \frac{(aq; q)_n}{(aq; q)_\nu}$, this becomes

$$(-1)^n a^{\frac{n(n-3)}{2}} q^{\frac{n(n+1)(2n-5)}{6}} (abcq; q^2)_n \prod_{k=1}^n \frac{(q; q)_{k-1} (aq; q)_k (bq; q)_{k-2}}{(abq^2; q)_{k+n-2}} \times {}_4\phi_3 \left(\begin{matrix} a^{\frac{1}{2}} c^{\frac{1}{2}} q^{\frac{1}{2}}, -a^{\frac{1}{2}} c^{\frac{1}{2}} q^{\frac{1}{2}}, q^{-n}, abq^n \\ a^{\frac{1}{2}} b^{\frac{1}{2}} c^{\frac{1}{2}} q^{\frac{1}{2}}, -a^{\frac{1}{2}} b^{\frac{1}{2}} c^{\frac{1}{2}} q^{\frac{1}{2}}, aq \end{matrix} ; q, q \right).$$

If we replace a by aq^r , and multiply the both sides with $\left(\frac{(aq; q)_r}{(abq^2; q)_r} \right)^n$, then we obtain (1.15). There are several ways to write (1.15) with the Askey-Wilson polynomials, and (1.16) follows from (1.15) using the definition (1.6). \square

Before we proceed to a proof of Corollary 1.3, we need the following contiguous relations for ${}_4\phi_3$.

Proposition 3.2. Let z, a, b, c, d, e, f, g and q be arbitrary parameters. Then we have

$$\begin{aligned} & {}_4\phi_3 \left(\begin{matrix} a, bq, c, d \\ e, f, g \end{matrix}; q, z \right) - {}_4\phi_3 \left(\begin{matrix} aq, b, c, d \\ e, f, g \end{matrix}; q, z \right) \\ &= \frac{z(b-a)(1-c)(1-d)}{(1-e)(1-f)(1-g)} {}_4\phi_3 \left(\begin{matrix} aq, bq, cq, dq \\ eq, fq, gq \end{matrix}; q, z \right), \end{aligned} \quad (3.2)$$

$$\begin{aligned} & (1-f)(a-e) {}_4\phi_3 \left(\begin{matrix} a, b, c, d \\ eq, f, g \end{matrix}; q, z \right) - (1-e)(a-f) {}_4\phi_3 \left(\begin{matrix} a, b, c, d \\ e, fq, g \end{matrix}; q, z \right) \\ &= (1-a)(f-e) {}_4\phi_3 \left(\begin{matrix} aq, b, c, d \\ eq, fq, g \end{matrix}; q, z \right), \end{aligned} \quad (3.3)$$

and

$$\begin{aligned} & (1-e)(1-f)(1-g) {}_4\phi_3 \left(\begin{matrix} a, b, c, d \\ e, f, g \end{matrix}; q, q \right) \\ &= c(1-e) \left(1 - \frac{f}{c}\right) \left(1 - \frac{g}{c}\right) {}_4\phi_3 \left(\begin{matrix} aq, bq, c, d \\ e, fq, gq \end{matrix}; q, q \right) \\ &+ d(1-c) \left(1 - \frac{e}{d}\right) \left(1 - \frac{fg}{cd}\right) {}_4\phi_3 \left(\begin{matrix} aq, bq, cq, d \\ eq, fq, gq \end{matrix}; q, q \right), \end{aligned} \quad (3.4)$$

where, in the last identity, we assume $abcdq = efg$ and $a = q^{-n}$ for some nonnegative integer n .

Proof. First, (3.2) and (3.3) are readily proven by direct computation. The last identity (3.4) is written as

$$\begin{aligned} & (1-e)(1-f)(1-g) {}_4\phi_3 \left(\begin{matrix} q^{-n}, \frac{efgq^{n-1}}{cd}, c, d \\ e, f, g \end{matrix}; q, q \right) \\ &= c(1-e) \left(1 - \frac{f}{c}\right) \left(1 - \frac{g}{c}\right) {}_4\phi_3 \left(\begin{matrix} q^{-n+1}, \frac{efgq^n}{cd}, c, d \\ e, fq, gq \end{matrix}; q, q \right) \\ &+ d(1-c) \left(1 - \frac{e}{d}\right) \left(1 - \frac{fg}{cd}\right) {}_4\phi_3 \left(\begin{matrix} q^{-n+1}, \frac{efgq^n}{cd}, cq, d \\ eq, fq, gq \end{matrix}; q, q \right), \end{aligned} \quad (3.5)$$

and can be proven by induction on n by using only (3.2). The details are left to the reader. \square

Remark 3.3. The contiguous relations (3.2) (resp. (3.3)) correspond to (3.2) (resp. (3.10)) in [9], meanwhile (3.3) can be written as a contiguous relation for ${}_8W_7$. In fact, if one uses Watson's transformation formula [4, (2.5.1)]

$${}_8W_7 \left(a; b, c, d, e, q^{-n}; q, \frac{a^2 q^{n+2}}{bcde} \right) = \frac{(aq, \frac{aq}{de}; q)_n}{(\frac{aq}{d}, \frac{aq}{e}; q)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, d, e, \frac{aq}{bc} \\ \frac{aq}{b}, \frac{aq}{c}, \frac{deq^{-n}}{a} \end{matrix}; q, q \right) \quad (3.6)$$

for a terminating very-well-poised ${}_8\phi_7$ series, where

$${}_{r+1}W_r(a_1; a_4, \dots, a_{r+1}; q, z) = {}_{r+1}\phi_r \left(\begin{matrix} a_1, qa_1^{\frac{1}{2}}, -qa_1^{\frac{1}{2}}, a_4, \dots, a_{r+1} \\ a_1^{\frac{1}{2}}, -a_1^{\frac{1}{2}}, \frac{qa_1}{a_4}, \dots, \frac{qa_1}{a_{r+1}} \end{matrix}; q, z \right), \quad (3.7)$$

then (3.4) is equivalent to

$$\begin{aligned} & (c-a)(d-aq)(e-aq)(b-aq^n) {}_8W_7 \left(a; b, cq, d, e, q^{-n}; q, \frac{a^2 q^{n+1}}{bcde} \right) \\ &= a(1-b)(1-aq)(de-aq)(1-cq^n) {}_8W_7 \left(aq; bq, cq, d, e, q^{-n+1}; q, \frac{a^2 q^{n+1}}{bcde} \right) \\ &+ (bc-a)(d-aq)(e-aq)(1-aq^n) {}_8W_7 \left(a; b, c, d, e, q^{-n+1}; q, \frac{a^2 q^{n+1}}{bcde} \right). \end{aligned} \quad (3.8)$$

The following proposition is crucial to derive Corollary 1.3 from Theorem 1.1.

Proposition 3.4. Let n be an integer, a , b and c be arbitrary parameters. Then we have

$$\begin{aligned} p_n(0; a, b, c, -c; q) &= (-1)^m a^m b^m c^{2m} q^{m(3m-1)} (-c^2; q^2)_m \\ &\times p_m(x_0; 1, q, ab, -a^{-1}b^{-1}c^{-2}q^{-4m+2}; q^2), \end{aligned} \quad (3.9)$$

if $n = 2m$ is even, and

$$\begin{aligned} p_n(0; a, b, c, -c; q) &= (-1)^{m+1} a^m b^{m+1} c^{2m} (1+ab^{-1}) q^{m(3m+1)} (-c^2; q^2)_{m+1} \\ &\times p_m(x_0; q, q^2, ab, -a^{-1}b^{-1}c^{-2}q^{-4m}; q^2), \end{aligned} \quad (3.10)$$

if $n = 2m + 1$ is odd, where $x_0 = -\frac{ab^{-1}+a^{-1}b}{2}$.

Proof. We proceed by induction on n . If $n = -1$ (resp. $n = 0$), then the both sides of (3.10) (resp. (3.9)) equals 0 (resp. 1). Hence (3.9) and (3.10) holds when $n = -1, 0$. Assume $n \geq 1$ and (3.9) and (3.10) hold when $n-1$ and $n-2$. If $n = 2m$, then

$$p_{2m}(0; a, b, c, -c; q)$$

by (1.6)

$$= \frac{(ab, ac, -ac; q)_{2m}}{a^{2m}} {}_4\phi_3 \left(\begin{matrix} q^{-2m}, -abc^2 q^{2m-1}, a, -a \\ ab, ac, -ac \end{matrix}; q, q \right)$$

by (3.2)

$$\begin{aligned}
&= \frac{(ab, ac, -ac; q)_{2m}}{a^{2m}} {}_4\phi_3 \left(\begin{matrix} q^{-2m+1}, -abc^2q^{2m-2}, a\iota, -a\iota \\ ab, ac, -ac \end{matrix}; q, q \right) \\
&\quad - (1+a^2)(1+abc^2q^{4m-2}) \frac{(abq, acq, -acq; q)_{2m-1}}{a^{2m}q^{2m-1}} \\
&\quad \times {}_4\phi_3 \left(\begin{matrix} q^{-2m+1}, -abc^2q^{2m-1}, aq\iota, -aq\iota \\ abq, acq, -acq \end{matrix}; q, q \right) \\
&= a^{-1}(1-abq^{2m-1})(1-acq^{2m-1})(1+acq^{2m-1}) p_{2m-1}(0; a, b, c, -c; q) \\
&\quad - a^{-1}(1+a^2)(1+abc^2q^{4m-2}) p_{2m-1}(0; aq, b, c, -c; q).
\end{aligned}$$

By the induction hypothesis (3.10), this equals

$$\begin{aligned}
&= (-1)^m a^{m-2} b^m c^{2m-2} q^{(m-1)(3m-2)} (-c^2; q^2)_m \\
&\times \left\{ (1-abq^{2m-1})(1-acq^{2m-1})(1+acq^{2m-1})(1+ab^{-1}) \right. \\
&\quad \times p_{m-1}(x_0; q, q^2, ab, -a^{-1}b^{-1}c^{-2}q^{-4m+4}; q^2) \\
&\quad - q^{m-1}(1+a^2)(1+abc^2q^{4m-2})(1+ab^{-1}q) \\
&\quad \times p_{m-1}(x_1; q, q^2, abq, -a^{-1}b^{-1}c^{-2}q^{-4m+3}; q^2) \left. \right\},
\end{aligned}$$

where $x_1 = -\frac{ab^{-1}q+a^{-1}bq^{-1}}{2}$. Hence, by (1.6) again, this becomes

$$\begin{aligned}
&= (-1)^m c^{2m} q^{m(3m-1)} (-c^2; q^2)_m (abq^2, abq^3, -c^{-2}q^{-4m+4}; q^2)_{m-1} \\
&\times \left\{ -a^2(1-abq)(1+a^{-1}b)(1-a^{-2}c^{-2}q^{-4m+2}) \right. \\
&\quad \times {}_4\phi_3 \left(\begin{matrix} q^{-2m+2}, -c^{-2}q^{-2m+3}, -a^2, -b^2 \\ abq, abq^2, -c^{-2}q^{-4m+4} \end{matrix}; q^2, q^2 \right) \\
&\quad - b^2(1+ab^{-1}q)(1+a^2)(1+a^{-1}b^{-1}c^{-2}q^{-4m+2}) \\
&\quad \times {}_4\phi_3 \left(\begin{matrix} q^{-2m+2}, -c^{-2}q^{-2m+3}, -a^2q^2, -b^2 \\ abq^3, abq^2, -c^{-2}q^{-4m+4} \end{matrix}; q^2, q^2 \right) \left. \right\}.
\end{aligned}$$

Using (3.5), we obtain

$$\begin{aligned}
&= (-1)^m c^{2m} q^{m(3m-1)} (-c^2, ab, abq, -c^{-2}q^{-4m+2}; q^2)_m \\
&\quad \times {}_4\phi_3 \left(\begin{matrix} q^{-2m}, -c^{-2}q^{-2m+1}, -a^2, -b^2 \\ abq, ab, -c^{-2}q^{-4m+2} \end{matrix}; q^2, q^2 \right)
\end{aligned}$$

by (1.6) again

$$\begin{aligned}
&= (-1)^m a^m b^m c^{2m} q^{m(3m-1)} (-c^2; q^2)_m \\
&\quad \times p_m(x_0; q, 1, ab, -a^{-1}b^{-1}c^{-2}q^{-4m+2}; q^2).
\end{aligned}$$

This proves (3.9) when $n = 2m$. It is well-known that the Askey-Wilson polynomials (1.6) are symmetric with respect to the parameters a, b, c and d . Meanwhile, the expression of the Askey-Wilson polynomials by the ${}_4\phi_3$ series apparently depends on the choice of the parameter a in (1.6). Note that we have to choose this parameter carefully in each step to apply the above contiguous relations. Similarly, when $n = 2m + 1$, (3.10) can be proven using the definition (1.6) of the Askey-Wilson polynomials, the contiguous relations (3.2), (3.3) and the induction hypothesis (3.9) for $n = 2m$. We omit the details for the reader. Hence we conclude that (3.9) and (3.10) hold for arbitrary m . \square

When $b = -a$, replacing c by b , one gets incidentally the following known result due to Andrews (see [4, (II.17)]).

Corollary 3.5.

$$\begin{aligned} p_n(0; a, -a, b, -b; q) \\ = \begin{cases} (-1)^m (q, -a^2, -b^2, a^2 b^2 q^{2m}; q^2)_m & \text{if } n = 2m, \\ 0 & \text{if } n = 2m + 1. \end{cases} \quad \square \end{aligned}$$

Proof of Corollary 1.3. When $n = 2m$ is even, if we apply (3.9) to (1.16) using $(b; q^2)_m \prod_{k=1}^{2m} (bq; q)_{k-2} = \prod_{k=1}^m \{(bq; q)_{2k-2}\}^2$ then we obtain (1.19) by straightforward computation. (1.18) is derived from (1.19) by (1.6) and using $(q; q^2)_m \prod_{k=1}^{2m} (q; q)_{k-1} = \prod_{k=1}^m \{(q; q)_{2k-1}\}^2$, $(aq^{r+1}; q^2)_m \prod_{k=1}^{2m} (aq; q)_{k+r+1} = \prod_{k=1}^m \{(aq; q)_{2k+r+1}\}^2$ and $\frac{(a^{-1}b^{-1}q^{1-4m-r}; q^2)_m}{\prod_{k=1}^{2m} (abq^2; q)_{k+2m+r-2}} = \frac{(-1)^m a^{-m} b^{-m} q^{-m(3m+r+1)}}{\prod_{k=1}^m \{(abq^2; q)_{2(k+m)+r-3}\}^2}$.

When $n = 2m + 1$ is even, if we apply (3.10) to (1.16) using

$$(b; q^2)_{m+1} \prod_{k=1}^{2m+1} (bq; q)_{k-2} = \prod_{k=1}^{m+1} (bq; q)_{2k-2} \cdot \prod_{k=1}^m (bq; q)_{2k-2}$$

then we obtain (1.21). Finally, (1.20) is obtained from (1.21) by using (1.6),

$$\begin{aligned} (a^{-1}b^{-1}q^{-4m-r}; q^2)_m &= (-1)^m a^{-m} b^{-m} q^{-m(3m+r+1)} (abq^{2m+r+2}; q^2)_m, \\ (q^3; q^2)_m \prod_{k=1}^{2m+1} (q; q)_{k-1} &= \frac{1}{1-q} \cdot \prod_{k=1}^{m+1} (q; q)_{2k-1} \prod_{k=1}^m (q; q)_{2k-1}, \\ (aq^{r+2}; q^2)_m \prod_{k=1}^{2m+1} (aq; q)_{k+r-1} &= \prod_{k=1}^{m+1} (aq; q)_{2k+r-2} \prod_{k=1}^m (aq; q)_{2k+r}, \\ \frac{(abq^{2m+r+2}; q^2)_m}{\prod_{k=1}^{2m+1} (abq^2; q)_{k+2m+r-1}} &= \frac{1}{\prod_{k=1}^{m+1} (abq^2; q)_{2k+2m+r-2} \prod_{k=1}^m (abq^2; q)_{2k+2m+r-2}}. \end{aligned}$$

This completes the proof of Corollary 1.3. \square

4 A quadratic relation

In this section we use the Desnanot-Jacobi adjoint matrix theorem to derive a quadratic relation between the Askey-Wilson polynomials for a special values of parameters.

First we recall the reader a well-known theorem for matrix. The following identity is known as the Desnanot-Jacobi adjoint matrix theorem [2, Theorem 3.12]

$$\det A_{[2,n-1]}^{[2,n-1]} \det A_{[n]}^{[n]} = \det A_{[n-1]}^{[n-1]} \det A_{[2,n]}^{[2,n]} - \det A_{[2,n]}^{[n-1]} \det A_{[n-1]}^{[2,n]}. \quad (4.1)$$

Let

$$D_n(a, b, c; q) = \det \left((q^{i-1} - cq^{j-1}) \frac{(aq; q)_{i+j-2}}{(abq^2; q)_{i+j-2}} \right)_{1 \leq i, j \leq n}$$

and apply (4.1) to this determinant. Then we obtain

$$\begin{aligned} D_n(a, b, c; q) D_{n-2}(aq^2, b, c; q) &= \frac{q(aq; q)_2}{(abq^2; q)_2} \cdot D_{n-1}(a, b, c; q) D_{n-1}(aq^2, b, c; q) \\ &- \frac{q(1-aq)^n(1-abq^3)^{n-2}}{(1-aq^2)^{n-2}(1-abq^2)^n} \cdot D_{n-1}(aq, b, cq; q) D_{n-1}(aq, b, cq^{-1}; q). \end{aligned} \quad (4.2)$$

Hence we can substitute (1.16) into (4.2). Then we obtain

$$\begin{aligned} &aq(1-q^{n-1})(1-bq^{n-2}) \cdot p_n(0; a^{\frac{1}{2}}c^{\frac{1}{2}}q^{\frac{1}{2}}i, -a^{\frac{1}{2}}c^{-\frac{1}{2}}q^{\frac{1}{2}}i, b^{\frac{1}{2}}i, -b^{\frac{1}{2}}i; q) \\ &\quad \times p_{n-2}(0; a^{\frac{1}{2}}c^{\frac{1}{2}}q^{\frac{3}{2}}i, -a^{\frac{1}{2}}c^{-\frac{1}{2}}q^{\frac{3}{2}}i, b^{\frac{1}{2}}i, -b^{\frac{1}{2}}i; q) \\ &= (1-aq^n)(1-abq^n) \cdot p_{n-1}(0; a^{\frac{1}{2}}c^{\frac{1}{2}}q^{\frac{1}{2}}i, -a^{\frac{1}{2}}c^{-\frac{1}{2}}q^{\frac{1}{2}}i, b^{\frac{1}{2}}i, -b^{\frac{1}{2}}i; q) \\ &\quad \times p_{n-1}(0; a^{\frac{1}{2}}c^{\frac{1}{2}}q^{\frac{3}{2}}i, -a^{\frac{1}{2}}c^{-\frac{1}{2}}q^{\frac{3}{2}}i, b^{\frac{1}{2}}i, -b^{\frac{1}{2}}i; q) \\ &- (1-aq)(1-abq^{2n-1}) \cdot p_{n-1}(0; a^{\frac{1}{2}}c^{\frac{1}{2}}q^{\frac{3}{2}}i, -a^{\frac{1}{2}}c^{-\frac{1}{2}}q^{\frac{1}{2}}i, b^{\frac{1}{2}}i, -b^{\frac{1}{2}}i; q) \\ &\quad \times p_{n-1}(0; a^{\frac{1}{2}}c^{\frac{1}{2}}q^{\frac{1}{2}}i, -a^{\frac{1}{2}}c^{-\frac{1}{2}}q^{\frac{3}{2}}i, b^{\frac{1}{2}}i, -b^{\frac{1}{2}}i; q). \end{aligned} \quad (4.3)$$

Replacing $a^{\frac{1}{2}}c^{\frac{1}{2}}q^{\frac{1}{2}}i$, $-a^{\frac{1}{2}}c^{-\frac{1}{2}}q^{\frac{1}{2}}i$ and $b^{\frac{1}{2}}i$ by a , b and c , respectively in (4.3), we obtain the following corollary.

Corollary 4.1. Let n be a positive integer and a , b , c and q parameters. Then we have

$$\begin{aligned} &ab(1-q^{n-1})(1+c^2q^{n-2})p_n(0; a, b, c, -c; q)p_{n-2}(0; aq, bq, c, -c; q) \\ &= (1-abq^{n-1})(1+abc^2q^{n-1})p_{n-1}(0; a, b, c, -c; q)p_{n-1}(0; aq, bq, c, -c; q) \\ &\quad - (1-ab)(1+abc^2q^{2n-2})p_{n-1}(0; aq, b, c, -c; q)p_{n-1}(0; a, bq, c, -c; q). \end{aligned} \quad (4.4)$$

In other word

$$\begin{aligned}
& abq(1 - q^{n-1})(1 + c^2q^{n-2}) {}_4\phi_3 \left(\begin{matrix} q^{-n}, -abc^2q^{n-1}, aq, -aq \\ ab, ac, -ac \end{matrix}; q, q \right) \\
& \quad \times {}_4\phi_3 \left(\begin{matrix} q^{-n+2}, -abc^2q^{n-1}, aq, -aq \\ abq^2, acq, -acq \end{matrix}; q, q \right) \\
& = (1 - abq^n)(1 + abc^2q^{n-1}) {}_4\phi_3 \left(\begin{matrix} q^{-n+1}, -abc^2q^{n-2}, aq, -aq \\ ab, ac, -ac \end{matrix}; q, q \right) \\
& \quad \times {}_4\phi_3 \left(\begin{matrix} q^{-n+1}, -abc^2q^n, aq, -aq \\ abq^2, acq, -acq \end{matrix}; q, q \right) \\
& \quad - (1 - abq)(1 + abc^2q^{2n-2}) {}_4\phi_3 \left(\begin{matrix} q^{-n+1}, -abc^2q^{n-1}, aq, -aq \\ abq, acq, -acq \end{matrix}; q, q \right) \\
& \quad \times {}_4\phi_3 \left(\begin{matrix} q^{-n+1}, -abc^2q^{n-1}, aq, -aq \\ abq, ac, -ac \end{matrix}; q, q \right). \tag{4.5}
\end{aligned}$$

Here we derive Corollary 4.1 as a corollary of Theorem 1.1. Note that, if one can prove the quadratic relation (4.4) or (4.5) directly, then he can prove Theorem 1.1 using the Desnanot-Jacobi adjoint matrix theorem (4.1). This was our first strategy to prove Theorem 1.1, but we have found it is not so easy to prove the quadratic relation. Hence we prove Theorem 2.1 which is more general, and then derive Theorem 1.1.

Finally, we state a conjecture which generalizes (4.4). We can observe that the following quadratic equation with two more parameters could hold concerning to the Askey-Wilson polynomials:

Conjecture 4.2. Let n be a positive integer and x, a, b, c, d and q parameters. Then we have

$$\begin{aligned}
& ab(1 - q^{n-1})(1 - cdq^{n-2})p_n(x; a, b, c, d; q)p_{n-2}(x; aq, bq, c, d; q) \\
& = (1 - abq^{n-1})(1 - abcdq^{n-1})p_{n-1}(x; a, b, c, d; q)p_{n-1}(x; aq, bq, c, d; q) \\
& \quad - (1 - ab)(1 - abcdq^{2n-2})p_{n-1}(x; aq, b, c, d; q)p_{n-1}(x; a, bq, c, d; q). \tag{4.6}
\end{aligned}$$

Concluding Remarks This conjecture may hint us there could exist a more general formula than Theorem 1.1. But it is not an easy task to find the appropriate entry of the determinant which gives this quadratic relation.

It is also an interesting problem to find a combinatorial application of Theorem 1.1. If $c = 0$, then (1.7) is the generating function of the Dyck paths with certain weights (see [6]). For $c = 1$, the Pfaffian (1.8) can enumerate certain reverse plane partitions (see [7]). We have been trying to find an application of Theorem 1.1 and it is not so easy to find an appropriate lattice path and its weights. It will be left for the future work.

Acknowledgments We are grateful to Professor Christian Krattenthaler for his encouragement at the initial stage of this project.

References

- [1] G. E. Andrews, R. Askey, R. Roy, *Special Functions*, Cambridge Univ. Press, (1999).
- [2] D. M. Bressoud, *Proofs and Confirmations: The Story of the Alternating-Sign Matrix Conjecture*, Cambridge University Press, 1997, 1999.
- [3] M. Ciucu and C. Krattenthaler, “The Interaction of a gap with a free boundary in a two dimensional dimer system”, *Comm. Math. Phys.*, **302** (2011) 253 – 289.
- [4] G. Gasper and M. Rahman, *Basic Hypergeometric Series (2nd ed.)*, Cambridge Univ. Press, (1990, 2004).
- [5] M. Ishikawa, H. Kawamuko and S. Okada, “A Pfaffian-Hafnian analogue of Borchardt’s identity”, *Electron. J. Combin.*, **12** (1), (2005) #N9.
- [6] M. Ishikawa, H. Tagawa and J. Zeng, “A q -analogue of Catalan Hankel determinants”, *RIMS Kôkyûroku Bessatsu*, **B11** (2009), 19–42.
- [7] M. Ishikawa, H. Tagawa and J. Zeng, “Pfaffian decomposition and a Pfaffian analogue of q -Catalan Hankel determinants”, [arXiv:1011.5941v1](https://arxiv.org/abs/1011.5941).
- [8] C. Krattenthaler, “Determinants of (generalized) Catalan numbers”, *J. Statist. Plann. Inference* **140** (2010), 2260–2270, [arXiv:0709.3044](https://arxiv.org/abs/0709.3044).
- [9] C. Krattenthaler, “A Systematic List of Two- and Three-term Contiguous Relations for Basic Hypergeometric Series”, available at <http://www.mat.univie.ac.at/~kratt/artikel/contrel.html>.
- [10] R. Koekoek, P. Lesky and R. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q -Analogues*, Springer-Verlag, (2000).
- [11] R. Koekoek and R. F. Swarttouw, “The Askey-scheme of hypergeometric orthogonal polynomials and its q -analogue”, Department of Technical Mathematics and Informations, Faculty of Information Technology and Systems, Deift University of Technology, Reports no. 98-17, (1998).
- [12] A. Lascoux, “Pfaffians and Representations of the Symmetric Groups”, *Acta Math. Sin.* **25** (2009), 1929–1950.
- [13] A. Lascoux, “Hankel Pfaffians, Discriminants and Kazhdan-Lusztig bases”, [arxiv:1103.497](https://arxiv.org/abs/1103.497).
- [14] M. Mehta and R. Wang, “Calculation of a Certain Determinant”, *Commun. Math. Phys.* **214** (2000), 227–232.

- [15] M. Nishizawa, “Evaluation of a certain q -determinant” *Linear Algebra Appl.* **343** (2002), 107–115.
- [16] R. Stanley, *Enumerative combinatorics, Volume I, II*, Cambridge University Press, 1997, 1999.
- [17] G. Viennot, “A combinatorial theory for general orthogonal polynomials with extensions and applications” *Lecture Notes in Mathematics* **1171** (1985) Springer-Verlag, 139-157, 1997, 1999.